

A STUDY
OF THE
SPECTRUM OF ZETA₁ URSAE MAJORIS

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE
UNIVERSITY OF MICHIGAN

By
LAURENCE HADLEY

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A STUDY OF THE SPECTRUM OF ζ_1 URSAE MAJORIS¹

By LAURENCE HADLEY

INTRODUCTION AND STATEMENT OF THE PROBLEM

As early as the middle of the seventeenth century ζ Ursae Majoris ($\alpha = 13^h 20^m$, $\delta = +55^\circ 23'$) was observed as a double star. This was the first discovery of its kind and was made by Riccioli in 1650. It was again observed as a visual double star in 1700 by Kirch. However, the position angle and distance of this pair were not measured until 1750, when they were recorded by Bradley as $143^\circ.1$ and $13''.88$, respectively. Many measures have been made and published since that time. The recorded values of the position angle seem to show a very slow relative motion which probably does not exceed $0^\circ.025$ per year. The distance between the two components, ζ_1 and ζ_2 , as given by the best observers for more than a half century, does not vary greatly on either side of 14.4 seconds of arc. So slow are the changes in these quantities that thousands of years will have passed before a single revolution will be complete.

The data to be found on page 399 of the *Astronomical Observatory of Yale Transactions*, Vol. 2, give $0''.021 \pm 0''.008$ as the most probable parallax of this star, from which it follows that its distance from the sun is approximately 155 light years and that the linear separation of the two components of this visual pair is not less than 64,000,000,000 miles and it may be much greater than this value, depending on the angle between the line of sight and the line joining the two stars.

Basing her calculation on a parallax of $0''.045$, Miss Clerke has estimated that Mizar "sends abroad thirty-eight times more light than our sun".² Since there is good reason to believe that the parallax is not greater than half of the value used by Miss Clerke, it follows that the light

giving power of this star is probably not less than one hundred and fifty times that of the sun.

Professor Stebbins gives 2.40 as the magnitude of ζ_1 Ursae Majoris. In the *Astrophysical Journal*, Vol. 39, page 475, he makes the following statement as a result of his investigation: "Observations throughout the spectroscopic period give no evidence of eclipses, nor of continuous variation. The light is constant within 0.02 or 0.03 mag."

It is an interesting fact that this second magnitude star should have been the first visual binary discovered and that the brighter of its two components, ζ_1 , should have been the first known spectroscopic binary. In making a study of the plates taken with an objective-prism spectrograph at the Harvard College Observatory, Miss Maury observed that the K line of ζ_1 Ursae Majoris appeared as a double line on certain plates, while on many others it was a well defined single line. The separation of the K line was so evident on certain plates that its reality could not be doubted. This discovery was made in 1889 and was the beginning of a new line of most fruitful investigation in Astronomy. The Harvard plates, though numerous, had been taken at irregular intervals and, as a result, the period of this doubling of the lines was not evident. It was erroneously placed at 52 days, thus making the time of a complete revolution 104 days. This period of 104 days is approximately five times the true period, which was first derived by Vogel. The Harvard staff found difficulty in their effort to harmonize the period of 104 days with the doubling of the lines as shown by the plates. At times the lines appeared as sharply defined single lines, when the period of 104 days would have called for a wide separation. However, it was left for Vogel to discover the true time of revolution of this spectroscopic pair and to announce a period which would satisfy the observations.

This star, ζ Ursae Majoris, is of peculiar interest in that the fainter component, ζ_2 , has been announced as showing variable radial velocity.

¹ A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan.

² Clerke, *Problems in Astrophysics*, page 290.

Frost and Lee³, of the Yerkes Observatory, have announced the variation as from -17 km. to $+10$ km., while Dr. Ludendorff⁴ has announced that the range shown by his plates is from -17.3 km. to -7.3 km. Nearly a hundred spectrograms have been measured and reduced by the writer. A frequency curve based on the velocities obtained from these measures conforms closely to the probability curve, thus indicating that the differences are due, largely at least, to accidental errors rather than to real changes, so far as plates made with a single prism instrument are concerned.

It is the purpose of this paper to give the results of a thorough study of ζ_1 Ursæ Majoris, the brighter of the two visual components. The results and conclusions will be based on velocities published by Dr. Ludendorff and those which were obtained from the measures of spectrograms⁵ made at the Detroit Observatory, Ann Arbor, Michigan. It is the aim in this discussion to treat the observations separately; to make a complete and independent solution for each component of ζ_1 Ursæ Majoris; and finally, to co-ordinate the results of the two sets of observations and of the two components, thus putting the subject upon a basis which will be as complete and definitive as possible.

HISTORICAL

*Harvard Observations.*⁶ As previously stated ζ_1 Ursæ Majoris was the first known spectroscopic binary. The first plate to show the K line distinctly double was made in 1886, but this fact was not observed until 1889. More than 150 plates of this star were made at the Harvard College Observatory between the years 1885 and 1889. These spectrograms were used primarily in connection with the work of the observatory on the classification of stellar spectra. However,

³ *Astronomische Nachrichten*, Vol. 177, page 171.

⁴ *Astronomische Nachrichten*, Vol. 177, page 9.

⁵ For complete description of the apparatus employed and the constants of the instruments see *Publications of the Detroit Observatory*, Vol. 1.

⁶ *Monthly Notices*, Vol. 50, page 296.

Harvard College Observatory Circular, No. 11.

Harvard College Observatory Annals, Vol. 26, page xvii.

Astrophysical Journal, Vol. 8, page 174.

some consideration was given to the doubling of the lines; to the meaning of this apparent change in the spectrum; to the probable period; and to the velocity in the line of sight, which would correspond to the relative displacement of the lines as observed. There was considerable uncertainty as to the period, but it was thought to be approximately 104 days. Professor Pickering derived the relative velocity of the two components of this spectroscopic binary. Using the displacements of the K line, he found the relative velocity to be 94 miles per second, and from the magnesium line, λ 4481, he placed the value at 102 miles per second. In this connection he says it was not probable that the plates were made at the time of a maximum displacement, hence the maximum velocity is certainly not less than 100 miles per second.⁷

Potsdam Observations. During the months of March and April, 1901, Dr. Ludendorff and Dr. Eberhard secured a series of plates of ζ_1 Ursæ Majoris with Spectrograph IV of the 33 cm. refractor of the Potsdam Observatory. Vogel undertook the measurement and reduction of these plates. The results of his work are given in Vol. XIII, page 324, of the *Astrophysical Journal*. The results are based on the relative velocities of one component with respect to the other and not on the velocities of each component with respect to the center of mass of the system. The measures of only 25 plates are included in Vogel's publication. When Dr. Ludendorff, at a later time, discussed this series of plates, he rejected 13 of the 25 used by Vogel as being of uncertain value, owing to the very small relative displacement of the lines. In Dr. Ludendorff's paper he states that the time recorded is that of the middle of the plate's exposure. Since the time as given by Vogel for these same plates is from 20 to 30 minutes earlier, it seems to be a reasonable supposition that he gave the time for the beginning of the exposure. This would introduce an appreciable error, particularly in the steepest part of the velocity curve. In consideration of this fact and the fact that the good plates are included in the data published by Dr. Ludendorff, which will be discussed in detail, nothing further will be

⁷ *American Journal of Science*, Third Series, Vol. 39, page 46.

done with these earlier results except to give the elements which Dr. Eberhard derived by the method of Lehmann-Filhés from Vogel's measures. The elements follow:

$$\begin{aligned}
 T_0 &= 1901, \text{ March } 28.60. \text{ (Relative motion in} \\
 &\quad \text{line of sight } = 0), \\
 T &= 1901, \text{ March } 28.88, \\
 \omega &= 101^\circ.3, \\
 e &= 0.502, \\
 \log \mu &= 9.4843, \\
 \mu &= 17''.476, \\
 P &= 20.6 \text{ days,} \\
 a \sin i &= 35 \text{ million km.,} \\
 m_1 + m_2 &= \frac{4 \cdot \odot}{\sin^3 i}.
 \end{aligned}$$

Dr. Ludendorff and Dr. Eberhard continued their observations in 1901 throughout the months of May and June. The work was taken up again in the years 1903, 1905, and 1906. In all, 118 spectrograms were secured. Of these 118 plates, the measures by Dr. Ludendorff of 59 are given in the *Astronomische Nachrichten*, Vol. 180, No. 4313, page 278. These measures have not been used for the determination of a complete set of elements. They were used for finding the center of mass velocity; the ratio of the masses; and for improving the period as announced by Vogel.

Since a full discussion of Dr. Ludendorff's measures will be taken up in this paper, the details of his work will be omitted and his final results stated. The period, he says, is correct to within a few thousandths of a day. The probable errors of the other quantities show what confidence may be placed in their determination.

$$\begin{aligned}
 P &= 20.536 \text{ days,} \\
 \gamma &= -12.6 \pm 0.49 \text{ km.,} \\
 \frac{m_1}{m_2} &= 1.014 \pm 0.018,
 \end{aligned}$$

where P is the period; γ the velocity of the center of mass of the system; and $\frac{m_1}{m_2}$ the ratio of the masses of the two components.

It is worthy of note at this point that the values of γ and $\frac{m_1}{m_2}$ are somewhat uncertain. The values given above were derived from the entire series of 59 plates. Dr. Ludendorff also divided the plates into two groups; the first, containing the

plates made in 1901, the second, those of 1903, 1905, and 1906. The first group contained 26 plates and the second 33. When these two groups were considered separately, unexpected differences were found as shown below:

1901.	1903-6.
$\gamma = -13.2 \text{ km.,}$	$\gamma = -12.2 \text{ km.,}$
$\frac{m_1}{m_2} = 0.980.$	$\frac{m_1}{m_2} = 1.035.$

These differences will be discussed later, but it may be well to note at this point that such change in the ratio of the masses would not be possible, and that the change in center of mass velocity seems to be improbable, unless there is a third body in the system, a resisting medium, or other disturbing factors.

DERIVATION OF ELEMENTS.

Potsdam Measures. The velocities referred to above, as derived by Dr. Ludendorff, will now be discussed in full. Table I contains the data as given in the *Astronomische Nachrichten*. Column (1) gives the number of the plate; column (2) gives the Greenwich mean time for the middle of the observation; column (3) gives the phase as counted from 1912 March 10.000, using the final value of the period; the next three columns contain data for Component 1, the first gives the number of lines measured, the second gives the velocity in the line of sight, and the third gives the error in the sense, observed minus computed; the next three columns refer to Component 2, giving the same data as was given for Component 1; the last column gives the weight to be assigned. It is difficult for anyone, other than he who measures the plates, to determine the weights which should be assigned. Hence, the writer adopted Dr. Ludendorff's assignment of weights and used the same plan in the reduction of his own measures. It is as follows. Unless there be good reason for an increase or decrease in the weight to be assigned to a plate, it is made equal to one-half of the sum of the number of lines measured on the two components. There are other methods of assigning weights which are better, but it seemed desirable to treat both sets of data exactly alike. Hence, the plan adopted by the observer at Potsdam was used in both cases.

THE SPECTRUM OF ZETA₁ URSAE MAJORIS

TABLE I. VELOCITIES

POTSDAM OBSERVATIONS.

NO. (1)	DATE, G. M. T. (2)	PHASE, (3)	COMPONENT 1.			COMPONENT 2.			WT. (10)
			N. (4)	V. (5)	O-C. (6)	N. (7)	V. (8)	O-C. (9)	
	1901	DAYS		KM.	KM.		KM.	KM.	
1	Mar. 24.383	0.989	3	+42	+68	5	-67	-8.3	4
2	26.393	2.999	4	+42	-3.7	3	-66	+7.6	3
3	29.293	5.899	0	..		1	+36	+1.5	0
4	30.333	6.939	4	-90	-4.2	3	+64	+3.8	3
5	Apr. 2.313	9.919	0	..		4	+44	+2.5	0
6	16.333	3.402	11	+44	-2.2	3	-78	-2.7	5
7	17.333	4.402	4	+36	+2.3	2	-59	+5.8	3
8	20.323	7.392	5	-88	-9.7	3	+47	-8.5	2
9	21.333	8.402	3	-76	-2.7	2	+38	-11.5	2
10	22.333	9.402	3	-63	-2.2	3	+46	+8.7	3
11	23.323	10.392	4	-52	-3.2	2	+22	-5.4	3
12	24.343	11.412	3	-40	-0.2	1	+16	-2.8	2
13	May 2.363	19.432	2	+23	+2.7	1	-48	-5.8	1
14	3.353	20.422	2	+22	-5.2	2	-53	-3.2	2
15	5.363	1.869	7	+46	+8.7	6	-68	-6.7	6
16	12.363	8.896	4	-67	-0.5	3	+46	+4.5	3
17	13.343	9.876	1	-62	-6.8	3	+34	+2.2	2
18	14.343	10.876	2	-54	-8.8	4	+20	-2.6	3
19	24.373	0.369	5	+35	+4.2	3	-60	-6.2	4
20	31.353	7.349	7	-85	-0.2	5	+56	-2.8	6
21	June 1.353	8.349	7	-72	+1.8	5	+48	-0.8	6
22	13.373	20.369	2	+38	+11.2	1	-47	+2.8	1
23	19.413	5.873	3	-65	-10.5	4	+41	+6.8	3
24	20.353	6.813	6	-87	-1.5	2	+56	-4.2	4
25	21.393	7.853	5	-82	-1.7	8	+50	-4.8	3
26	23.363	9.823	3	-60	-4.2	4	+32	-0.8	3
	1903								
27	Apr. 28.432	6.190	8	-69	+1.5	8	+51	+2.4	8
28	30.432	8.190	2	-75	+1.5	2	+58	+7.4	2
29	May 13.313	0.534	5	+40	+8.2	5	-48	+7.3	5
30	14.373	1.594	4	+46	+6.8	6	-56	+7.7	5
31	20.323	7.544	5	-75	+8.2	9	+50	-7.2	7
32	22.323	9.544	4	-60	-1.4	3	+39	+3.5	3
33	23.323	10.544	3	-57	-8.4	2	+28	+2.2	2
	1905								
34	Apr. 20.353	10.335	4	-53	-2.7	4	+25	-2.8	4
35	30.333	20.315	2	+21	-5.8	2	-47	+2.3	2
36	May 3.353	2.799	7	+40	-5.0	7	-67	+5.8	7
37	4.363	3.809	8	+36	-8.2	7	-78	-3.2	7

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TABLE I. VELOCITIES.

POTSDAM OBSERVATIONS—CONT'D.

NO.	DATE, G. M. T.		PHASE.	COMPONENT 1.			COMPONENT 2.			WT.
				N.	V.	O-C.	N.	V.	O-C.	
(1)	(2)		(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	1905		DAYS		KM.	KM.		KM.	KM.	
38	May	9.343	8.789	1	—70	—1.6	1	+39	—5.8	1*
39		25.343	4.252	5	+43	+5.5	5	—75	—5.5	5
40		27.343	6.252	3	—70	+2.5	3	+44	—5.6	3
41		28.343	7.252	4	—83	+2.8	5	+65	+6.7	4
42		29.343	8.252	6	—77	—1.2	7	+55	+4.6	6
43		30.363	9.272	4	—60	+3.3	3	+39	+0.2	3
44		31.363	10.272	9	—49	+2.3	9	+26	—2.6	9
45	June	19.423	8.796	3	—64	+4.6	5	+43	—0.7	4
	1906									
46	Mar.	29.403	4.266	4	+41	+3.2	1	—36	+33.3	0†
47	Apr.	1.373	7.236	2	—83	+2.7	2	—62	+2.8	2
48		3.373	9.236	1	—73	—9.8	1	+28	—10.6	1*
49		15.353	0.679	2	+33	+0.2	3	—63	—6.7	2
50		17.303	2.629	3	+50	—4.8	1	—75	—3.6	2
51		21.363	6.689	3	—89	—3.6	5	+58	—2.0	4
52		23.313	8.639	3	—68	+2.5	6	+51	+5.0	4
53		24.343	9.669	4	—55	+2.5	5	+31	—3.3	4
54	May	5.333	0.123	7	+20	—9.3	11	—51	+0.8	9
55		6.323	1.113	8	+37	+1.2	6	—65	—5.4	7
56		7.343	2.133	6	+41	—1.2	5	—66	+2.2	5
57		8.323	3.113	3	+53	+6.8	3	—64	+10.2	3
58		13.333	8.123	6	—82	—6.0	7	+59	+8.5	6
59		14.323	9.113	4	—59	+5.5	3	+44	+3.7	3

* Under exposure.

† Only 1 line on Comp. 2.

Normal places. The final velocities as given in Table I were combined into sixteen normal places with the phase as the basis of grouping. Care was taken that the limits of phase for each normal place be so small that the velocity curve over that interval would be essentially straight. The weight assigned to each place was determined by the number of good, measurable lines on all of the plates used in making up the normal place. The phases as given in column (2) have been reduced to the sun. Column (3) gives the limits within which the phases of the plates fall which were used in forming each normal place. The velocities of the two components are given; also

the residuals in the sense, observed minus computed, based on the final elements.

The normal velocities given in Table II are plotted in Plate VI. The exact positions are shown by the centers of the small circles.

From the phases which had been reduced to the sun and the velocities corrected for curvature as given in Table II, velocity curves were plotted. From these curves preliminary elements for each component were derived by use of the Forty-five Degree Chordal Method.⁸ The two components were carried entirely through, separately, Component 1 having been taken up first. But before

⁸ *Astrophysical Journal*, Vol. 28, page 212.

taking up either component it is desirable to settle the question of the period.

Period. In order to obtain an accurate value for the period, Ludendorff's velocities were plotted with P taken equal to 20.536 days which was known already to be a good value. The Ann Arbor velocities were similarly plotted and the curves transferred to tracing cloth which was then fitted, as nearly as could be, to the curves from Ludendorff's measures. With the curves in the best agreement, the time-axes were compared at six different, well defined points and the average of the six determinations gave $\Delta T = +0.07$ days.

The mean time for the Potsdam observations was taken as 1903.5 and for the Ann Arbor observations, 1912.6. The time interval between these two dates is 9.1 years which is approximately 161 periods. From this it follows that

$$\Delta P = \frac{+0.07}{161} = +0.00044 \text{ days.}$$

This correction is much smaller than might have been expected since Ludendorff stated that his determination, $P = 20.536$, was correct within a few thousandths of a day. By applying the correction which was found, we have $P = 20.53644$ days as the true period. From the method of its determination and the large time interval over which it extended, we shall adopt this as final and no further correction for this element will be sought in connection with the least square solution. The period having been accurately found, the value of μ , the mean daily motion, was easily derived.

$$\mu = 17^{\circ}.5298.$$

Since μ and P are mutually dependent, the values of these two quantities will be accepted without further correction. Since $\delta\mu$ will not be needed in the least-square formulæ, only five unknowns will occur and the computation will be greatly simplified.

TABLE II. NORMAL PLACES.

POTSDAM OBSERVATIONS.

NO. (1)	PHASE. (2)	LIMITS OF PHASE. (3)	WT. (4)	COMPONENT 1.		COMPONENT 2.	
				VELOCITY. (5)	O-C. (6)	VELOCITY. (7)	O-C. (8)
1	6.135	5.87 to 6.26	0.56	-69.11	+0.29	+48.10	+1.36
2	6.802	6.68 to 6.94	0.44	-88.93	-3.46	+59.08	-0.95
3	7.439	7.23 to 7.56	1.00	-82.06	+2.14	+55.32	-2.80
4	8.250	8.12 to 8.41	0.88	-76.73	-1.35	+52.96	+2.96
5	8.768	8.63 to 8.90	0.48	-66.60	+2.30	+46.04	+1.87
6	9.260	9.11 to 9.41	0.40	-61.90	+0.96	+41.50	+2.76
7	9.711	9.54 to 9.92	0.48	-58.67	-1.11	+33.75	-0.18
8	10.336	10.27 to 10.55	0.72	-51.28	-0.64	+25.33	-2.29
9	11.144	10.87 to 11.42	0.16	-47.00	-4.56	+18.00	-2.02
10	20.212	19.43 to 20.43	0.24	+24.50	-1.37	-49.27	-0.98
11	20.828	20.65 to 21.07	0.72	+28.89	-1.27	-52.17	+0.98
12	21.543	21.21 to 21.65	0.52	+37.92	+2.82	-65.31	-6.37
13	22.412	22.13 to 22.67	0.64	+44.44	+3.70	-63.62	+2.40
14	23.415	23.16 to 23.65	0.60	+44.39	-1.08	-67.73	+5.47
15	24.175	23.93 to 24.35	0.48	+39.65	-5.29	-78.35	-3.31
16	24.844	24.79 to 24.94	0.32	+40.49	+4.43	-69.10	-1.55

Component 1. Velocity curves having been drawn with as much accuracy as was possible, the following set of preliminary elements was derived:

PRELIMINARY ELEMENTS (COMPONENT 1).

$$\begin{aligned} P &= 20.53644 \text{ days,} \\ \mu &= 17^{\circ}.5298, \\ T &= 5.600 \text{ days from 1912 March 10.000.} \\ e &= 0.5440, \\ \omega &= 101^{\circ} 12'.2, \\ K &= 68.0 \text{ km.,} \\ V'_{\infty} &= -18.0 \text{ km.,} \\ \gamma &= -10.82 \text{ km.} \end{aligned}$$

From these elements an ephemeris was computed and the residuals found in the sense, observed minus computed. The agreement being sufficiently close between observation and computation, a least square solution was undertaken. The sixteen normal places gave sixteen equations of condition. The usual method of treatment gave the five normal equations which follow. The notation is that of Schlesinger in his paper published in the *Allegheny Observatory Publications*, Vol. 1, No. 6.

NORMAL EQUATIONS.

$$\begin{aligned} 8.64\Gamma - 0.870\kappa - 3.248\pi - 0.712\epsilon - 0.271\tau + 8.794 &= 0, \\ + 5.730\kappa + 0.383\pi + 1.209\epsilon + 0.014\tau + 7.264 &= 0, \\ + 2.910\pi + 0.730\epsilon + 0.715\tau + 0.620 &= 0, \\ + 0.945\epsilon + 0.258\tau + 0.478 &= 0, \\ + 0.305\tau + 0.806 &= 0. \end{aligned}$$

The five normal equations were treated after the method given in Vol. 1 of Chauvenet's *Practical Astronomy*. A solution of these equations gave the following values:

$$\begin{aligned} \tau &= +0.4844, \\ \epsilon &= +2.6638, \\ \pi &= -3.1655, \\ \kappa &= -1.9489, \\ \Gamma &= -2.1693. \end{aligned}$$

To check the accuracy of the numerical work, the values of the unknowns were substituted in the normal equations. The agreement was found

to be satisfactory, the largest of the residuals being 0.005 while the average value was 0.003. From these values of the unknowns the corrections to be applied to the elements were obtained. The corrections and the corrected elements follow:

CORRECTIONS.

$$\begin{aligned} \delta K &= -1.949 \text{ km.,} \\ \delta \omega &= +2^{\circ} 40'.0, \\ \delta e &= -0.01248, \\ \delta T &= +0.00577 \text{ days,} \\ \delta \gamma &= -0.851 \text{ km.} \end{aligned}$$

CORRECTED ELEMENTS.

$$\begin{aligned} T &= 5.60577 \text{ days,} \\ e &= 0.53152, \\ \omega &= 103^{\circ} 52'.2, \\ K &= 66.051 \text{ km.,} \\ \gamma &= -11.671 \text{ km.} \end{aligned}$$

The residuals recorded in column 6 of Table II were found by comparison of the observed velocities with an ephemeris computed from the corrected elements. The differences between these residuals and those found from the normal equations were so small that the second order terms neglected in making a least square solution were evidently inappreciable, hence a repetition of the least square solution was not necessary.

It is now possible to find the quantity, $a \sin i$, by use of the formula,

$$a \sin i = [4.13833] (1 - e^2)^{1/2} K . P.$$

Substituting in the equation we obtain the following value:

$$a \sin i = 15,800,000 \text{ km.}$$

The probable errors given in connection with certain of the elements were computed by the usual method. The value for the mass, m_1 , cannot be determined at this point in the solution, but is given here even though derived later. The

THE SPECTRUM OF ZETA₁ URSAE MAJORIS

final elements for Component 1 derived from Ludendorff's measures follow:

FINAL ELEMENTS. (COMPONENT 1).

$$\begin{aligned}
 P &= 20.53644 \text{ days,} \\
 \mu &= 17^\circ.5298, \\
 T &= 5.6058 \pm 0.0700 \text{ days from} \\
 &\quad 1912, \text{ March } 10.000, \\
 e &= 0.53152 \pm 0.01024, \\
 \omega &= 103^\circ 52'.2 \pm 2^\circ 47'.9, \\
 K &= 66.05 \pm 0.77 \text{ km.,} \\
 \gamma &= -11.67 \text{ km.,} \\
 a \sin i &= 15,800,000 \text{ km.,} \\
 m_1 &= \frac{1.57 \odot}{\sin^3 i}.
 \end{aligned}$$

The probable errors in ω and T indicate some uncertainty in the values of these quantities. This was to be expected since, in forming the elimination equations, the quantity [*ee*. 4] is very small. This quantity, [*ee*. 4], is a function of the coefficients found in the normal equations and since it nearly vanishes owing to the peculiarities of the normal places and the velocity curve, the value of τ and hence of δT are uncertain. The value of ω changes with the value of T , therefore, it is uncertain also.

The least square solution having been completed, the following results may be stated:

1. The sum of the weighted squares of the residuals for the sixteen normal places was reduced from 87.48 to 53.52.
2. The probable error of a normal place of weight unity was found to be ± 1.49 km.
3. The residuals recorded in column 6 of Table I were scaled from the final velocity curve, Plate VI. These residuals gave ± 3.5 km. as the probable error of the average plate and ± 2.3 km. as the probable error of the best plates of the series.

Next in order should come a like treatment of the measures of Component 2. While the discussion is more brief, the details of the process were identically the same as those used in Component 1.

Component 2. As in the preceding case, preliminary elements were derived and residuals found by a comparison of the observed and the computed velocities for each of the sixteen normal places.

PRELIMINARY ELEMENTS. (COMPONENT 2).

$$\begin{aligned}
 P &= 20.53644 \text{ days,} \\
 \mu &= 17^\circ.5298, \\
 T &= 5.56 \text{ days from} \\
 &\quad 1912, \text{ March } 10.000, \\
 e &= 0.5410, \\
 \omega &= 282^\circ 6'.4, \\
 K &= 67.0 \text{ km.,} \\
 V'_0 &= -5.0 \text{ km.,} \\
 \gamma &= -12.61 \text{ km.}
 \end{aligned}$$

The normal equations derived from the sixteen equations of condition took the following form:

NORMAL EQUATIONS.

$$\begin{aligned}
 8.64\Gamma + 0.813\kappa + 2.459\pi + 0.836e + 0.297\tau + 4.712 &= 0, \\
 5.737\kappa + 0.489\pi + 1.197e + 0.060\tau - 0.912 &= 0, \\
 2.903\pi + 0.765e + 0.704\tau - 2.062 &= 0, \\
 0.794e + 0.247\tau + 1.459 &= 0, \\
 0.287\tau - 0.458 &= 0.
 \end{aligned}$$

As indicated in connection with Component 1, the elimination equations were formed from these normal equations. Again the value of [*ee*. 4] was small enough to indicate some uncertainty in the values of T and ω .

The five normal equations gave the following as the values of the unknowns:

$$\begin{aligned}
 \tau &= +0.8046, \\
 e &= -5.2975, \\
 \pi &= +2.4510, \\
 \kappa &= +1.1694, \\
 \Gamma &= -0.8679.
 \end{aligned}$$

The accuracy of the numerical work was shown by the small residuals which were obtained when the values of the unknowns were substituted in the normal equations, the average value for the five equations being 0.005, while the largest was only 0.007. The corrections and the corrected elements follow:

CORRECTIONS.

$$\begin{aligned}
 \delta K &= +1.169 \text{ km.,} \\
 \delta \omega &= -2^\circ 5'.8, \\
 \delta e &= +0.02530, \\
 \delta T &= +0.00983, \\
 \delta \gamma &= -0.0596 \text{ km.}
 \end{aligned}$$

FIRST CORRECTED ELEMENTS.

$$\begin{aligned}
 K &= 68.169 \text{ km.,} \\
 \omega &= 280^\circ 0'.6, \\
 e &= 0.56630, \\
 T &= 5.56983 \text{ days from} \\
 &\quad 1912, \text{ March } 10.000, \\
 \gamma &= -12.670 \text{ km.}
 \end{aligned}$$

The residuals (observed minus computed velocities) were found from the normal equations and also from an ephemeris, the corrected elements being used. The differences for eight of the sixteen normal places exceeded 0.1 and ranged in value from 0.11 to 0.31. To make sure that the second order terms were not affecting the results, a second least square solution was made in which the corrected elements were taken as a beginning. The normal equations follow:

NORMAL EQUATIONS (SECOND SOLUTION).

$$\begin{aligned} 8.64\Gamma + 0.879\kappa + 3.596\pi + 0.658\epsilon + 0.318\tau + 1.958 &= 0, \\ 5.567\kappa + 0.228\pi + 1.272\epsilon + 0.002\tau - 0.876 &= 0, \\ 3.073\pi + 0.716\epsilon + 0.651\tau - 0.686 &= 0, \\ 0.936\epsilon + 0.238\tau - 1.217 &= 0, \\ 0.246\tau + 0.066 &= 0. \end{aligned}$$

This set of equations gave the following as the values of the unknown:

$$\begin{aligned} \tau &= -10.8148, \\ \epsilon &= +2.8932, \\ \pi &= +3.6645, \\ \kappa &= -0.4082, \\ \Gamma &= -1.5328. \end{aligned}$$

As in the other solution, the value of the quantity, [ee. 4], was small. To check the numerical computation, the values just derived were substituted in the normal equations. The average residual obtained from the five equations was 0.005, which showed complete agreement between the values of the unknowns and the equations.

The corrections and the corrected elements resulting from the second least square solution are given below.

CORRECTIONS.

$$\begin{aligned} \delta K &= -0.4082 \text{ km.}, \\ \delta \omega &= -3^\circ 4'.8, \\ \delta e &= -0.01304, \\ \delta T &= -0.11832 \text{ days}, \\ \delta \gamma &= +0.706 \text{ km.} \end{aligned}$$

CORRECTED ELEMENTS.

$$\begin{aligned} K &= 67.761 \text{ km.}, \\ \omega &= 276^\circ 55'.8, \\ e &= 0.55326, \\ T &= 5.45151 \text{ days}, \\ \gamma &= -11.964 \text{ km.} \end{aligned}$$

The agreement between residuals by the normal equations and by ephemeris was satisfactory, thus indicating that second order terms were negligible. Hence, a third solution would yield unimportant changes.

As in the case of Component 1, the quantity $a \sin i$ was found. Its value follows:

$$a \sin i = 15,940,000 \text{ km.}$$

It is now possible to find the masses of the two components in terms of the mass of the sun and the angle i . The two values follow:

$$\begin{aligned} m_1 &= \frac{1.57 \odot}{\sin^3 i}, \\ m_2 &= \frac{1.46 \odot}{\sin^3 i}. \end{aligned}$$

The value of $\sin^3 i$ cannot be greater than unity and hence the numerators of these expressions represent the minimum values. However, in finding the ratio of the masses this indeterminate quantity cancels out and the following value is obtained:

$$\frac{m_1}{m_2} = 1.079.$$

As the final elements for Component 2, derived from the Ludendorff measures, we have the following:

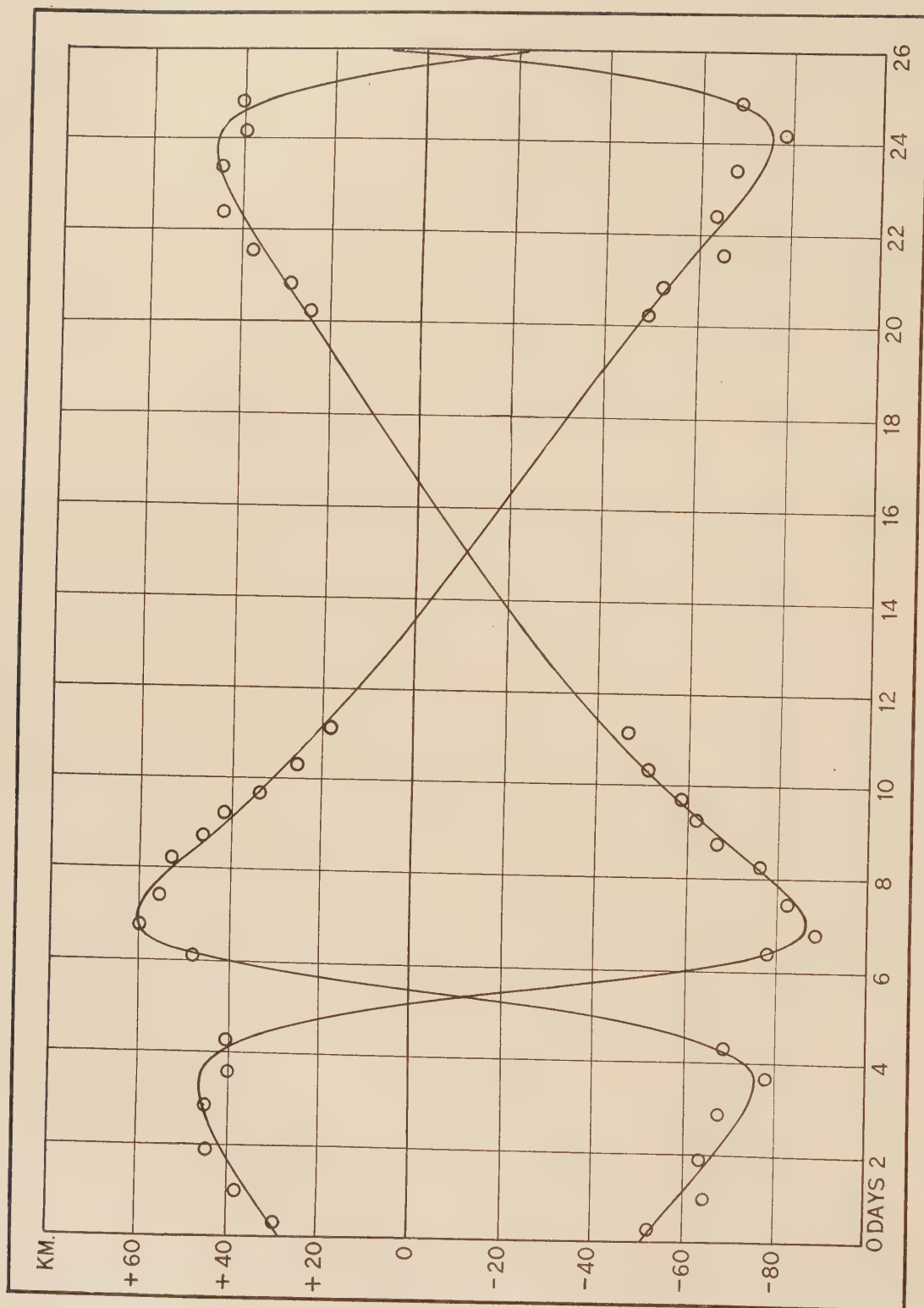
FINAL ELEMENTS (COMPONENT 2).

$$\begin{aligned} P &= 20.53644 \text{ days}, \\ \mu &= 17^\circ 52' 9'', \\ T &= 5.4515 \pm 0.0835 \text{ days from} \\ &\quad 1912, \text{ March } 10.000, \\ e &= 0.55326 \pm 0.01241, \\ \omega &= 276^\circ 55'.8 \pm 2^\circ 17'.5, \\ K &= 67.76 \pm 0.98 \text{ km.}, \\ \gamma &= -11.96 \text{ km.}, \\ a \sin i &= 15,940,000 \text{ km.}, \\ m_2 &= \frac{1.46 \odot}{\sin^3 i}, \end{aligned}$$

$$\frac{m_1}{m_2} = 1.079.$$

The uncertainty in the determination of T and ω is again evident from the probable errors.

Having arrived at a set of elements which give satisfactory agreement between observed and

PLATE VI. VELOCITY CURVES OF ζ_1 URSAE MAJORIS. VELOCITIES BY LUDENDORFF

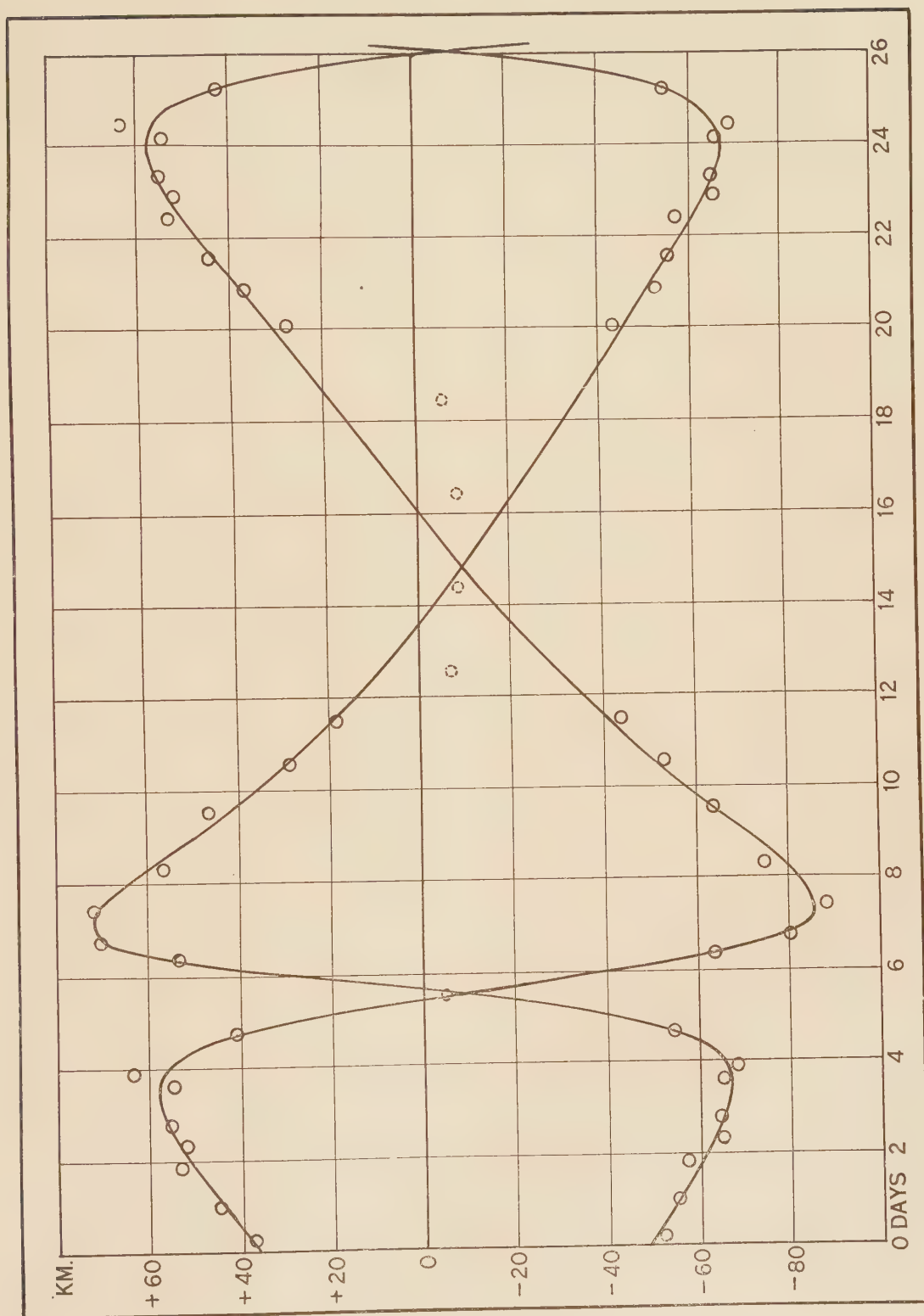


PLATE VII. VELOCITY CURVES OF ζ_1 URSAE MAJORIS. ANN ARBOR VELOCITIES

computed velocities, the following points of interest should be noted:

1. The sum of the weighted squares of the residuals was reduced from 99.66 to 83.12 by the first least square solution and to 75.83 by the second.

2. The probable error of a normal place of weight unity was found to be ± 1.77 km.

3. The residuals recorded in column 9 of Table I were scaled from the final velocity curve, Plate VI. From these residuals, the probable error of an average plate was found to be ± 3.7 km. and for the best plates of the series it was found to be ± 2.4 km.

Velocity Curves. The velocity curves shown in Plate VI were plotted from an ephemeris computed from the final elements for each of the two components as derived from the Potsdam measures. Points were located at frequent intervals throughout the entire length of the curves so that they might give an accurate representation of the velocities in all parts of the orbits. Beginning at the left border of the plate, the upper curve belongs to Component 1 and the lower one to Component 2.

This completes the discussion of Dr. Ludendorff's measures with the exception of certain questions arising in connection with the ratio of the masses and the center of mass velocity. These two points will be taken up and discussed later in this paper.

ANN ARBOR OBSERVATIONS

Observations of ζ_1 Ursæ Majoris were begun at the Detroit Observatory March 10, 1912, and were continued until August 27, of the same year. During the following year, a few plates were made in order to strengthen the weak parts of the velocity curve. In all 137 spectrograms were made. Of this number, 128 were measurable, 90 plates showing the lines measurably separated and 38 showing single lines. In general, lantern slide plates were used. Of the 128 measurable plates, 122 were of this variety, while 6 were Seed 23. The average time of exposure for the lantern slide plates was 9.1 minutes for 108 plates. Fourteen lantern slide plates were made with a very narrow slit with an average time of exposure equal to 35 minutes. The 6 Seed 23

plates required an exposure of 5.4 minutes on the average. While much time might have been saved by using the more rapid plates, the coarser grain made measuring much more difficult and in some cases, impossible. The lantern slide plates should be used on this and similar binaries in preference to more rapid plates. It is an interesting fact that the spectrograms made with the one prism instrument of the Detroit Observatory gave points on the velocity curve almost as near the cross-points of the curves as did the plates made with the three prism instrument at Potsdam.

Standard of Dispersion. The writer has previously measured and reduced a hundred plates of γ Lyræ. For use in that piece of work, a standard was obtained by taking the average screw reading for each of twenty-eight well defined titanium lines of the comparison spectrum as measured on ten of the best plates of the series. Three good lines, well distributed over the region of the spectrum to be used, were selected for the determination of the constants of the corrected Hartmann interpolation formula. This set of constants, and hence the same standard of dispersion, were used in the reduction of the plates of ζ_1 Ursæ Majoris.

Measures. The spectrum of ζ_1 Ursæ Majoris is usually classified as belonging to Class Ap, meaning that the star is of division A of the Harvard classification with certain peculiarities. The measures have been considered as difficult by each astronomer who has worked on the plates of this star. Vogel says, "When the spectra are more strongly displaced with reference to one another, most of the lines which appear double, become so weak that the measurement of their separation is rendered very difficult. On some plates, in fact, it was found possible to measure only the Mg line, λ 4481. It is to be noted that the fine lines of the spectra of Class Ia2 are only obtained when the exposure time is exactly correct, and the plate correctly developed. In general, the measures are to be classed as difficult, either on account of their excessive fineness or, in case of the Mg and H lines, on account of the too great width and diffuseness of the lines." All other measurers have experienced the difficulties set forth in the quotation from Vogel. The uncertainty of the measures is shown by the

size of the probable error of a single plate and was also very evident from the lack of agreement in the measures of the displacement of the various lines on a single plate.

Identification of lines. Of the 30 lines used in this series of plates, all except 5 were identified with certainty so that the corresponding wave-lengths were obtainable from the tables giving such data. The wave-lengths for the five unidentified lines were found as will be explained later.

Preliminary velocities. Having identified as many lines as possible, the next process was the determination of the velocities resulting from the displacements shown by the known lines. The wave-lengths adopted were taken from recognized authorities as will be seen by referring to Table III. Using the corrected Hartmann interpolation formula and the constants previously found, the screw reading for zero displacement of each star line was computed and also the corresponding factor necessary to transform displacements in terms of screw readings into velocities in km. per second. The velocity, as given from the displacement of each known measured line on every plate, was then found. The arithmetical mean of the velocities as shown by the lines of each plate was taken as the velocity for that plate unless the lines were clearly of unequal weight. By this means, a complete set of preliminary velocities was determined.

Unknown lines. These preliminary velocities and the proportionality factors of the unknown lines for the purpose of changing screw readings into velocities having been determined, it was not difficult to reverse the process and find the wave-lengths of the lines so that they would give the same velocity as that shown by the known lines of the plate. This was done for each unknown line for all of the plates on which it was measured and a weighted mean of the results taken as the wave-length of that particular unidentified line. The results are shown in Table III. The accuracy of these wave-lengths and confidence to be placed in their determination is shown by their probable errors and by the number of measures.

Use of the newly determined lines. The probable errors in the values of the wave-lengths,

which were found, indicate such accuracy as would lead to a better determination of the velocities from the measures of the separate plates if these new lines were included. Hence, the velocities were computed, use being made of the displacements of these new lines. That the values were improved was very evident from the closer agreement between the velocities as determined from successive plates made on the same night.

Final velocities. One other correction was necessary before the velocities could be accepted as final. The residuals for each line on all of the plates, which showed measurable separation, were found. These residuals came from a comparison of the velocity as given by the line itself with the velocity as given by the plate. A correction to be applied to each line was then found which would reduce to zero the algebraic sum of the residuals of that line for all of the plates on which it was measured. These corrections produced noticeably better agreement between observed velocities and those computed from the final elements.

Table III. This table contains data concerning the lines used in this investigation. The first column gives the assumed wave-length of the line. The second column gives the authority. R stands for Rowland, K for Kayser, L for Lockyer, RHC for Professor Ralph H. Curtiss of this observatory in his paper on β Lyræ, L H for the value as determined by the writer of this paper. The third column gives the intensity on a scale of 1 to 20. The fourth column gives the element, if it is known. The fifth column gives the wave-length in ζ_1 Ursæ Majoris, from which the final velocities were obtained. Columns 6, 7, and 8 give the number of times the lines were measured on plates showing separation of the components.

Following the table is to be found the computed wave-length for each of the unknown lines, the probable error, and the number of measures on which the determination depended. The probable errors do not, however, take into account the very slight inaccuracies affecting the determination of the wave-lengths by the corrected Hartmann interpolation formula.

Ann Arbor velocities. The results of the measures of the plates made at this Observatory are given in Tables IV and V. Table IV contains

THE SPECTRUM OF ZETA₁ URSAE MAJORIS

TABLE III. WAVE-LENGTHS.

ASSUMED WAVE-LENGTH.	AUTHORITY.	INTENSITY.	ELEMENT.	WAVE-LENGTH IN ζ ₁ URSAE MAJORIS.	NUMBER OF MEASURES.		
					COMP. 1	COMP. 2	TOTAL.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
3905.670	R	2	Si	3905.724	9	10	19
3913.640	K	2	Ti	3913.605	15	13	28
3933.825	R	10	Ca	3933.839	27	27	54
4005.730	RHC	2	?	4005.627	21	22	43
4045.975	R	3	Fe	4045.989	60	58	118
4063.770*	LH ¹	1	?	4063.773	42	43	85
4067.30	L	1	Enh. Ni.	4067.397	21	19	40
4071.908	R	1	Fe	4071.983	18	18	36
4077.89	L	2	Enh. Sr.	4077.928	41	40	81
4101.927	R	20	Hδ	4101.946	1	1	2
4128.204	RHC	1	Si	4128.336	16	13	29
4132.139	K	1	Van.	4132.455	10	12	22
4136.869*	LH ²	1	?	4136.826	9	8	17
4144.038	R	1	Fe	4143.992	31	28	59
4163.882*	LH ³	1	?	4163.877	6	6	12
4173.765	RHC	1	?	4173.748	36	38	74
4179.070	RHC	1	?	4179.146	7	5	12
4215.680	K	1	Sr	4215.725	43	42	85
4233.543*	LH ⁴	1	?	4233.505	39	39	78
4290.382	K	1	Ti	4290.327	22	26	48
4308.072*	LH ⁵	1	?	4308.114	17	15	32
4325.939	K	2	Fe	4325.799	10	10	20
4340.634	R	20	Hγ	4340.662	3	3	6
4352.42†	K	2	Sb		..	..	..
4443.99	L	1	Enh. Ti	4444.031	*11	11	22
4481.397	RHC	6	Mg	4481.462	46	45	91
4534.171	L	2	Enh. Ti	4534.220	3	0	3
4549.808†	R	2	Ti & Co		..	..	..
4572.473†	K	1	Ce		..	..	..
4861.527†	R	20	Hβ		..	..	..

* LH¹ λ 4063.770 ± 0.012, 71 measures.* LH² λ 4136.869 ± 0.040, 10 measures.* LH³ λ 4163.882 ± 0.035, 10 measures.* LH⁴ λ 4233.543 ± 0.019, 54 measures.* LH⁵ λ 4308.072 ± 0.039, 24 measures.

† This line was measured only on plates showing lines unresolved.

data coming from the plates which showed the lines of the two spectra measurably separated and Table V, the data coming from the plates in which the lines were not resolved and hence had to be measured as single lines. The different columns of these tables give the same data as the corresponding columns in the table of Luden-

dorff's measures. The initials in column (2) are those of the observer at the telescope; Prof. Curtiss, Mr. Mellor, and the writer. All of the measures were made by the writer. The measuring engine was carefully tested for screw error, which was found to be inappreciable. Hence, no correction was needed for that source of error.

THE SPECTRUM OF ZETA₁ URSAE MAJORIS

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TABLE IV. VELOCITIES.

ANN ARBOR OBSERVATIONS.

PLA. NO.	OBS.	DATE G. M. T.	PHASE.	COMPONENT 1			COMPONENT 2			WT. (11)
				N (5)	V (6)	O-C (7)	N (8)	V (9)	O-C (10)	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1912										
319	H	March 10.760	0.760	5	+ 53.2	+ 10.2	5	- 52.0	+ 1.5	5
321	H	10.786	0.786	3	+ 49.8	+ 6.6	3	- 59.9	- 6.0	3
333	M	12.792	2.792	8	+ 64.3	+ 8.0	7	- 49.7	+ 15.3	7
334	M	12.808	2.808	11	+ 48.5	- 7.8	8	- 76.4	- 11.4	9
358	M	16.847	6.847	9	- 73.8	+ 7.5	9	+ 76.6	+ 6.1	9
359	H	16.912	6.912	11	- 75.9	+ 6.1	11	+ 79.8	+ 9.3	11
412	H	29.809	19.809	5	+ 31.5	+ 1.9	4	- 31.1	+ 12.4	4
413	M	29.819	19.819	1	+ 21.0	- 8.6	1	- 44.5	- 1.0	1
421	M	Apr. 5.800	6.264	6	- 51.0	+ 7.0	5	+ 54.9	+ 7.9	5
422	H	5.811	6.275	5	- 61.2	- 3.2	3	+ 29.7	- 17.3	4
435	M	7.754	8.218	9	- 65.9	+ 14.1	10	+ 52.5	+ 1.0	9
436	M	7.764	8.228	3	- 84.5	- 4.5	3	+ 54.7	+ 3.2	3
460	M	23.750	3.677	8	+ 61.2	+ 3.8	8	- 55.1	+ 11.0	8
461	H	23.756	3.683	7	+ 54.6	- 2.8	7	- 69.5	- 3.4	7
472	C	26.708	6.635	8	- 70.2	+ 4.8	8	+ 72.6	+ 6.4	8
473	M	26.712	6.639	7	- 78.8	- 3.8	7	+ 60.6	- 5.6	7
475	M	26.738	6.665	9	- 88.1	- 13.1	9	+ 58.6	- 7.6	9
531	M	May 10.631	0.022	7	+ 38.2	+ 2.7	7	- 43.1	+ 5.5	7
533	M	10.671	0.062	4	+ 44.5	+ 7.5	3	- 47.6	+ 1.6	3
535	M	14.589	3.980	7	+ 53.4	- 2.4	7	- 73.2	- 8.2	7
536	M	14.594	3.985+	8	+ 66.1	+ 10.3	8	- 73.6	- 8.6	8
555	M	17.578	6.969	5	- 91.4	- 8.4	4	+ 59.7	- 11.5	4
556	M	17.585	6.976	3	- 95.3	- 12.3	3	+ 72.7	+ 1.5	3
568	C	19.672	9.063	7	- 68.1	+ 1.4	6	+ 57.9	+ 9.8	6
607	C	30.617	20.008	2	+ 20.4	- 0.9	2	- 45.2	- 0.2	2
608	C	30.624	20.015	2	+ 21.1	- 0.2	3	- 51.3	- 6.3	2
620	M	31.600	0.454	11	+ 30.0	- 3.2	7	- 52.7	- 1.1	9
621	M	31.606	0.460	6	+ 37.7	- 1.5	4	- 65.6	- 14.0	5
635	C	June 2.591	2.445	8	+ 35.1	- 18.8	8	- 63.2	+ 0.5	8
636	C	2.596	2.450	5	+ 46.0	- 7.9	7	- 64.6	- 1.1	6
657	M	4.627	4.481	5	+ 39.2	- 8.8	6	- 72.0	- 13.0	5
658	M	4.634	4.488	5	+ 30.9	- 17.1	5	- 66.0	- 7.0	5
668	H	6.580	6.434	10	- 66.9	+ 0.6	8	+ 53.0	- 5.0	9
669	H	6.594	6.448	4	- 63.9	+ 3.6	4	+ 55.1	- 3.0	4
670	H	6.610	6.464	9	- 65.1	+ 2.4	11	+ 58.2	+ 0.2	10
681	M	7.603	7.457	18	- 88.2	- 3.0	18	+ 73.1	+ 2.3	18
682	M	7.609	7.463	21	- 89.4	- 4.2	20	+ 71.6	+ 0.8	20
688	M	8.594	8.448	15	- 77.7	- 0.2	14	+ 54.6	- 3.6	14
689	M	8.601	8.455	15	- 72.6	+ 4.9	15	+ 61.5	+ 3.3	15
706	H	9.593	9.447	8	- 58.8	+ 5.4	7	+ 43.6	+ 0.2	7

THE SPECTRUM OF ZETA₁ URSAE MAJORIS

TABLE IV. VELOCITIES.

ANN ARBOR OBSERVATIONS—CONT'D.

PLA. NO.	OBS.	DATE		PHASE.	COMPONENT 1			COMPONENT 2			WT.
		G. M. T.			N	V	O-C	N	V	O-C	
(1)	(2)	(3)		(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1912											
707	H	June 9.606		9.460	6	-61.2	+ 3.0	6	+44.8	+ 1.4	6
718	C	10.617		10.471	5	-49.7	+ 1.6	5	+35.5	+ 5.3	5
719	C	10.639		10.493	7	-58.0	- 6.7	7	+24.3	- 5.9	7
731	M	11.617		11.471	6	-45.1	- 5.1	6	+ 9.2	-10.8	3
732	M	11.631		11.485	3	-42.7	- 2.8	3	+28.0	+ 7.8	3
749	M	21.669		0.987	6	+34.6	+ 9.1	5	-51.7	+ 3.6	5
750	M	21.677		0.995	6	+43.2	- 0.5	5	-66.1	-10.8	5
758	M	22.588		1.906	1	+46.1	- 4.2	1	-64.9	- 4.5	1
759	M	22.597		1.915	6	+47.7	- 2.6	7	-49.3	+11.1	6
773	M	28.619		7.937	2	-80.8	+ 2.2	2	+58.7	+ 3.9	2
774	M	28.624		7.942	4	-77.2	+ 5.0	4	+52.5	- 2.3	4
782	H	30.594		9.912	5	-64.4	+ 7.6	7	+45.3	+ 4.3	6
783	H	30.603		9.921	5	-64.2	+ 7.8	5	+43.5	+ 2.5	5
845	C	July 11.613		0.394	9	+33.1	- 6.1	10	-48.0	+ 3.5	9
846	C	11.625		0.406	7	+38.1	- 1.1	6	-62.2	-10.7	6
853	H	13.594		2.375	7	+62.0	+ 8.3	7	-69.9	- 6.7	4
854	H	13.608		2.389	5	+63.2	+ 9.5	5	-65.4	- 2.2	5
863	H	14.593		3.374	6	+37.7	-20.3	6	-73.2	- 5.7	6
864	H	14.606		3.387	8	+53.2	- 4.8	8	-60.8	- 3.2	8
873	H	17.638		6.419	14	-60.5	- 3.0	14	+61.0	+ 3.9	14
884	H	18.588		7.369	10	-80.9	+ 4.3	10	+76.1	+ 4.1	10
887	C	18.648		7.429	4	-93.9	- 8.7	4	+56.5	-15.5	4
926	H	31.577		20.359	3	+31.6	- 2.9	3	-48.4	- 0.7	3
927	H	31.636		20.417	3	+35.1	+ 0.6	3	-43.9	+ 3.8	3
936	C	Aug. 1.600		0.845	7	+43.1	+ 0.6	7	-48.7	+ 5.6	7
939	C	2.591		1.836	8	+42.8	- 7.0	7	-60.5	- 0.3	7
940	H	2.501		1.846	11	+60.0	+10.2	11	-52.3	+ 7.9	11
941	H	2.623		1.868	9	+58.9	+ 9.1	9	-63.7	- 3.5	9
949	H	4.580		3.825	8	+70.5	+13.5	9	-58.8	+ 8.9	8
950	H	4.588		3.833	5	+66.5	+ 9.5	5	-66.9	+ 0.8	5
951	H	4.597		3.842	7	+67.2	+10.2	8	-64.8	+ 2.9	7
952	H	4.603		3.848	8	+58.7	+ 1.7	8	-72.7	- 5.0	8
959	H	11.584		10.829	4	-47.5	+ 0.2	5	+29.4	+ 2.6	4
999	H	21.565		0.273	5	+32.2	- 6.5	6	-53.8	- 2.8	5
1000	H	21.586		0.294	5	+45.0	+ 6.3	6	-47.5	+ 3.5	5
1009	C	22.623		1.331	3	+40.1	- 5.1	4	-62.7	- 6.7	3
1010	C	22.647		1.355	6	+50.0	+ 4.8	6	-52.6	+ 3.4	6
1015	C	23.565		2.273	10	+51.6	- 1.8	10	-67.6	- 4.6	10
1016	H	23.590		2.298	9	+60.9	+ 7.5	9	-65.8	- 2.8	9
1017	H	23.619		2.327	11	+52.6	- 0.8	11	-61.7	+ 1.3	11

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TABLE IV. VELOCITIES.

ANN ARBOR OBSERVATIONS—CONT'D.

PLA. NO.	OBS.	DATE G. M. T.	PHASE.	COMPONENT 1			COMPONENT 2			WT.
				N	V	O-C	N	V	O-C	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
1912										
1030	H	Aug. 27.567	6.275	6	— 55.0	+ 5.2	6	+ 48.2	— 2.1	6
1031	H	27.600	6.308	7	— 58.2	+ 2.0	7	+ 46.4	— 3.9	7
1913										
2272	H	Aug. 29.597	3.650	17	+ 51.1	— 6.1	18	— 64.0	+ 2.0	17
2273	H	29.657	3.710	11	+ 60.5	+ 3.3	12	— 66.1	— 0.1	11
2274	H	29.662	3.715	11	+ 58.4	+ 1.2	10	— 64.3	+ 1.7	10
2278	H	30.613	4.666	16	+ 44.3	+ 0.3	16	— 54.5	+ 0.5	16
2279	H	30.639	4.692	14	+ 50.1	+ 6.6	15	— 61.9	— 6.9	14
2280	H	30.663	4.716	18	+ 47.7	+ 4.7	19	— 58.5	— 4.0	18
2281	H	30.675	4.728	14	+ 38.8	— 3.7	14	— 49.0	+ 5.0	14
2282	H	30.688	4.741	8	+ 31.9	— 10.1	6	— 41.0	+ 12.0	7

Normal Places. The data derived from plates showing the lines measurably separated will be discussed first. The observations were combined into sixteen normal places with phase as the basis of grouping. Care was taken that the limits of phase for each place be small enough for the velocity curves to be essentially straight over that interval. However, the correction for curvature was applied even though very small. The weight assigned was based on the number of measurable lines on all of the plates used in making up the normal place. The phases given in column (2) of Table VI are based on the final period and have been reduced to the sun. The values given in column (3) are the limits of phase within which all of the plates of that normal place fall. The velocities of the two components are given; also, the errors in the sense, observed minus computed, the final elements being used in determining the computed velocities.

The normal velocities, given in Table VI, are shown by the small circles as plotted in Plate VII.

Component 1. From the data of Table VI, a velocity curve was plotted and the following set of preliminary elements found.

PRELIMINARY ELEMENTS (COMPONENT 1).

$$\begin{aligned}
 P &= 20.53644 \text{ days,} \\
 \mu &= 17^{\circ}.5298, \\
 T &= 5.83 \text{ days from} \\
 &\quad 1912, \text{ March } 10.000, \\
 e &= 0.5224, \\
 \omega &= 103^{\circ} 18'.2, \\
 K &= 72.85 \text{ km.,} \\
 V_0' &= -13.35 \text{ km.,} \\
 \gamma &= -4.1 \text{ km.}
 \end{aligned}$$

An ephemeris having been computed and from it the residuals obtained, a least square solution was undertaken. The usual method of treatment gave the following

NORMAL EQUATIONS.

$$\begin{aligned}
 8.10\Gamma + 1.536\kappa - 0.485\pi - 0.185\epsilon + 0.659\tau + 5.332 &= 0, \\
 + 6.101\kappa - 0.297\pi - 0.096\epsilon - 0.109\tau + 8.715 &= 0, \\
 + 1.999\pi - 0.053\epsilon + 0.833\tau - 5.362 &= 0, \\
 + 0.894\epsilon + 0.037\tau - 1.020 &= 0, \\
 + 0.511\tau - 2.126 &= 0.
 \end{aligned}$$

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TABLE V. VELOCITIES.

ANN ARBOR OBSERVATIONS.

PLATE NO.	OBS.	DATE G. M. T.	PHASE.	SINGLE LINES.			WT.
				N	V	RESIDUALS.	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
		1912					
377	C	Mar. 22.827	12.827	16	— 3.5	+ 3.8	16
378	C	22.833	12.833	21	— 3.5	+ 3.8	21
452	H	Apr. 15.724	16.188	21	— 8.4	— 1.1	21
453	H	15.733	16.197	23	— 5.9	+ 1.4	23
495	H	May 2.655	12.582	26	— 3.0	+ 4.3	26
496	H	2.661	12.588	23	— 8.1	— 0.8	23
512	M	7.681	17.608	17	+ 2.7	+ 10.0	17
513	M	7.687	17.614	11	— 5.9	+ 1.4	11
517	H	8.669	18.596	24	— 2.6	+ 4.7	24
518	H	8.678	18.605	19	— 13.4	— 6.1	19
573	H	22.666	12.057	23	— 9.1	— 1.8	23
574	H	22.676	12.067	27	— 16.7	— 9.4	27
580	M	24.680	14.071	24	— 12.5	— 5.2	24
581	M	24.686	14.077	22	— 11.2	— 3.9	22
595	M	25.742	15.133	16	— 1.4	+ 5.9	16
596	M	25.749	15.140	25	— 5.5	+ 1.8	25
602	C	27.642	17.033	23	— 8.5	— 1.2	23
603	C	27.652	17.043	23	— 7.9	— 0.6	23
734	H	June 12.646	12.500	15	— 4.4	+ 2.9	15
735	H	12.653	12.507	17	— 5.9	+ 1.4	17
763	M	25.588	4.906	5	— 8.8	— 1.5	5
807	H	July 3.649	12.967	18	— 10.3	— 3.0	18
808	H	3.677	12.995	22	— 1.8	+ 5.5	22
811	M	6.653	15.971	19	— 13.3	— 6.0	19
812	M	6.675	15.993	17	— 14.5	— 7.2	17
825	M	9.625	18.943	6	— 2.6	+ 4.7	6
826	M	9.635	18.953	5	— 4.5	+ 2.8	5
902	H	25.598	14.379	6	— 3.6	+ 3.7	6
903	H	25.628	14.409	11	— 12.7	— 5.4	11
911	M	27.610	16.391	16	— 6.2	+ 1.1	16
912	M	27.622	16.403	16	— 6.1	+ 1.2	16
918	C	29.624	18.405	7	— 6.0	+ 1.3	7
919	C	29.638	18.419	8	— 3.9	+ 3.4	8
972	H	Aug. 14.580	13.825	13	— 3.3	+ 4.0	13
973	H	14.603	13.848	14	— 13.6	— 6.3	14
1020	H	26.580	5.288	7	— 7.9	— 0.6	7
1021	H	26.611	5.319	7	— 2.3	+ 5.0	7
1022	H	26.647	5.355	6	— 0.5	+ 6.8	6

These equations gave the following values of the unknowns:

$$\begin{aligned}\tau &= +1.048, \\ \epsilon &= +1.009, \\ \pi &= +2.004, \\ \kappa &= -1.202, \\ \Gamma &= +0.372.\end{aligned}$$

CORRECTED ELEMENTS (COMPONENT 1).

$$\begin{aligned}K &= 71.648 \text{ km.}, \\ \omega &= 101^\circ 43'.6, \\ e &= 0.51784, \\ T &= 5.84258 \text{ days from} \\ &\quad 1912, \text{ March } 10.000, \\ \gamma &= -6.221 \text{ km.}\end{aligned}$$

That these results are correct was shown by the small residuals obtained when the values of the unknowns were substituted in the normal equations. The largest of the residuals was 0.005 and the average of the five was 0.003.

The normal equations having been satisfied, the corrections to the elements and the corrected elements were found to be:

CORRECTIONS.

$$\begin{aligned}\delta K &= -1.202 \text{ km.}, \\ \delta \omega &= -1^\circ 34'.6, \\ \delta e &= -0.00456, \\ \delta T &= +0.01258 \text{ days}, \\ \delta \gamma &= -1.611 \text{ km.}\end{aligned}$$

The residuals as found from the normal equations and from the ephemeris as computed from these corrected elements were in satisfactory agreement. In no case was the difference larger than 0.08 km., while the average value was slightly less than 0.04 km. From this, it was evident that the second order terms neglected in the least square formulæ were negligible and hence a second least square solution was unnecessary.

As in the preceding cases, the formula for $a \sin i$ was applied. The value follows:

$$a \sin i = 17,310,000 \text{ km.}$$

When Component 2 had been carried through to this point, it was possible to find the mass of

TABLE VI. NORMAL PLACES.

ANN ARBOR OBSERVATIONS.

NO.	PHASE.	LIMITS OF PHASE.	WT.	COMPONENT 1		COMPONENT 2	
				VELOCITY.	O-C.	VELOCITY.	O-C.
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1	6.380	6.26 to 6.47	0.75	-63.40	+1.05	+53.61	-0.97
2	6.784	6.63 to 6.98	0.65	-80.06	-0.73	+70.33	+1.72
3	7.440	7.36 to 7.47	0.66	-88.03	-2.68	+71.84	+0.99
4	8.327	7.93 to 8.46	0.60	-74.41	+3.84	+56.43	-3.38
5	9.545	9.06 to 9.93	0.38	-63.58	-0.66	+46.94	+4.68
6	10.570	10.47 to 10.83	0.20	-52.78	-2.24	+29.08	-0.37
7	11.479	11.47 to 11.49	0.04	-43.90	-3.35	+18.60	-0.94
8	20.050	19.80 to 20.42	0.08	+27.95	-3.79	-42.80	+2.66
9	20.853	20.55 to 21.00	0.62	+37.27	-0.86	-52.21	-1.43
10	21.539	21.29 to 21.90	0.43	+44.89	+1.32	-55.10	+0.14
11	22.400	22.37 to 22.45	0.43	+53.59	+3.45	-56.85	+3.69
12	22.889	22.80 to 22.99	0.67	+52.14	-1.35	-65.03	-1.83
13	23.337	23.32 to 23.35	0.20	+55.41	-0.62	-64.72	+0.45
14	24.156	23.91 to 24.26	0.85	+54.43	-3.32	-65.35	+0.94
15	24.427	24.36 to 24.53	0.54	+63.70	+7.01	-68.43	-3.14
16	25.194	24.91 to 25.28	1.00	+42.94	-0.99	-53.94	+0.15

each component in terms of i and the mass of the sun. For Component 1 we have

$$m_1 = \frac{1.83 \odot}{\sin^3 i}.$$

Bringing together the data which have been derived, we have the following

FINAL ELEMENTS (COMPONENT 1).

$$\begin{aligned} P &= 20.53644 \text{ days,} \\ \mu &= 17^\circ.5298, \\ T &= 5.84258 \pm 0.07721 \text{ days from} \\ &\quad 1912, \text{ March } 10.000, \\ e &= 0.51784 \pm 0.00812, \\ \omega &= 101^\circ 43'.6 \pm 2^\circ 24'.9, \\ K &= 71.65 \pm 0.69 \text{ km.,} \\ \gamma &= -6.22 \text{ km.,} \\ a \sin i &= 17,310,000 \text{ km.,} \\ m_1 &= \frac{1.83 \odot}{\sin^3 i}. \end{aligned}$$

The probable errors of ω and T show about the same degree of uncertainty in the determination of these quantities as was found in the reduction of Ludendorff's measures. This was to have been expected since the quantity [*ce*. 4] almost vanished as before.

Points to be noted in connection with this curve and the corresponding elements are the following:

1. The sum of the weighted squares of the residuals was reduced by the least square solution from 90.86 to 62.04.

2. The probable error of a normal place of weight unity was found to be ± 1.60 km.

3. The residuals recorded in column (7) of Table IV were scaled from the velocity curve which was plotted from an ephemeris based on the final elements. See Plate VII. From these residuals, the probable error of an average plate was found to be ± 4.6 km. and that of the best plates of the series was found to be ± 2.8 km.

Component 2. From the data of Table VI, a velocity curve was plotted and the following set of preliminary elements found for Component 2.

PRELIMINARY ELEMENTS (COMPONENT 2)

$$\begin{aligned} P &= 20.53644 \text{ days,} \\ \mu &= 17^\circ.5298, \\ T &= 5.88 \text{ days from} \\ &\quad 1912, \text{ March } 10.000, \\ e &= 0.5420, \\ \omega &= 287^\circ 14'.6, \\ K &= 70.00 \text{ km.,} \\ V_0' &= +2.60 \text{ km.,} \\ \gamma &= -8.62 \text{ km.} \end{aligned}$$

By means of an ephemeris from these elements, the residuals were found for use in a least square solution. The sixteen equations of condition gave the following normal equations:

NORMAL EQUATIONS.

$$\begin{aligned} 8.10\Gamma - 1.630\kappa + 0.681\pi + 0.390\epsilon - 0.538\tau + 2.287 &= 0, \\ + 6.131\kappa - 0.831\pi - 0.028\epsilon - 0.071\tau + 3.942 &= 0, \\ + 1.969\pi + 0.037\epsilon + 0.746\tau - 1.458 &= 0, \\ + 0.864\epsilon + 0.054\tau + 0.196 &= 0, \\ + 0.434\tau - 1.102 &= 0. \end{aligned}$$

A solution of these equations gave the values of the unknowns:

$$\begin{aligned} \tau &= +5.672, \\ \epsilon &= -0.587, \\ \pi &= -1.770, \\ \kappa &= -0.791, \\ \Gamma &= +0.113. \end{aligned}$$

The usual test for accuracy in the computation was applied and satisfactory agreement found, the largest residual being 0.008 and the average of the five, 0.004. The corrections and the corrected elements follow:

CORRECTIONS.

$$\begin{aligned} \delta K &= -0.791 \text{ km.,} \\ \delta \omega &= +1^\circ 27'.9, \\ \delta e &= +0.00268, \\ \delta T &= +0.06611 \text{ days,} \\ \delta \gamma &= -0.732 \text{ km.} \end{aligned}$$

CORRECTED ELEMENTS (COMPONENT 2).

$$\begin{aligned} K &= 69.209 \text{ km.,} \\ \omega &= 288^\circ 42'.5, \\ e &= 0.54468, \\ T &= 5.94611 \text{ days from} \\ &\quad 1912, \text{ March } 10.000, \\ \gamma &= -9.352 \text{ km.} \end{aligned}$$

In only one case did the difference between the residuals as given by the normal equations and by the ephemeris exceed 0.1 km. and the value in that case was only 0.12 km. The average value of the sixteen differences was less than 0.04. A second least square solution was considered unnecessary, since it would yield no changes of importance.

As in the preceding cases, the value of the quantity $a \sin i$ was found.

$$a \sin i = 16,390,000 \text{ km.}$$

The formulæ for finding the masses were applied with the following results:

$$m_1 = \frac{1.83 \odot}{\sin^3 i},$$

$$m_2 = \frac{1.70 \odot}{\sin^3 i}.$$

A comparison of the two masses gives the following value for their ratio:

$$\frac{m_1}{m_2} = 1.025.$$

Bringing together the data concerning Component 2, we have the following

FINAL ELEMENTS (COMPONENT 2).

$$P = 20.53644 \text{ days,}$$

$$\mu = 17^\circ.5298,$$

$$T = 5.94611 \pm 0.05825 \text{ days from} \\ 1912, \text{ March } 10.000,$$

$$e = 0.54468 \pm 0.00681,$$

$$\omega = 288^\circ 42'.5 \pm 1^\circ 58'.6,$$

$$K = 69.21 \pm 0.56 \text{ km.,}$$

$$\gamma = -9.352 \text{ km.,}$$

$$a \sin i = 16,390,000 \text{ km.,}$$

$$m_2 = \frac{1.79 \odot}{\sin^3 i},$$

$$\frac{m_1}{m_2} = 1.025.$$

In connection with these elements and the corresponding velocity curve, the following points are of interest:

1. The sum of the weighted squares of the residuals was reduced from 41.41 to 40.92. The amount of this reduction is inconsiderable, but

the fact should be noted that the sum of the weighted squares from the preliminary elements was less than the final sum after the application of the method of least squares in each of the other cases, indicating that the preliminary elements in this case satisfied the observed velocities exceptionally well.

2. The probable error of a normal place of weight unity was found to be ± 1.30 km.

3. The residuals recorded in column (10) of Table IV were scaled from the final velocity curve as plotted from the corrected elements. See Plate VII. From these residuals the probable error of the average plate was found to be ± 4.2 km., while that of the best plates of the series was ± 2.5 km.

Velocity curves. The velocity curves shown in Plate VII were plotted from an ephemeris computed from the final elements for each of the two components as derived from the Ann Arbor measures. Points were located at frequent intervals throughout the entire length of the curves so that they might give an accurate representation of the velocities in all parts of the orbits. Beginning at the left border of Plate VII, the upper curve belongs to Component 1 and the lower curve to Component 2.

Ratio of the masses and the center of mass velocity. Since there was an apparent lack of agreement between the values determined for each of these quantities and since Ludendorff found a difference in the values as determined at different epochs, it seemed desirable to give these elements further consideration. A formula is easily derived which gives a mathematical relation between the following quantities:

V_1 = a velocity for component 1,

V_2 = the corresponding velocity for component 2,

γ' = an assumed value for the center of mass velocity,

$\Delta\gamma$ = correction to be applied to the assumed center of mass velocity,

$$\gamma = \gamma' + \Delta\gamma,$$

$$\frac{m_1}{m_2} = 1 + x, \text{ defines the quantity, } x.$$

If the assumed value of the center of mass velocity be near the true value and if the ratio of the masses be not far from unity, then $\Delta\gamma$ and x

are both small quantities. These conditions were both satisfied in the case under consideration. The formula follows and is exact except for the omission of the second order term, $\frac{1}{2}x \cdot \Delta\gamma$ which may be neglected without appreciable error.

$$\Delta\gamma - \frac{1}{2}(V_1 - \gamma') \cdot x = \frac{1}{2}(V_1 + V_2) - \gamma'.$$

Ann Arbor velocities. γ' having been assumed as -7.50 km., each pair of corresponding velocities when substituted in the equation gave an equation of condition. By the usual process these sixteen equations of condition were reduced to two normal equations with $\Delta\gamma$ and x as the unknowns. From a solution of the normal equations, the following values were found:

$$x = +0.032 \pm 0.011,$$

$$\Delta\gamma = +1.44 \pm 0.35 \text{ km.}$$

From which were obtained the ratio of the masses and the center of mass velocity:

$$\gamma = -6.06 \pm 0.35 \text{ km.,}$$

$$\frac{m_1}{m_2} = 1.032 \pm 0.011.$$

The equation given above is not symmetrical in V_1 and V_2 as is evident from the left member. The analogous equation was derived and applied, giving results in agreement with the values just found.

At this point, it should be noted that this value of γ is less than 1 km. different from the mean of the velocities obtained from nearly a hundred spectrograms of ζ_2 Ursæ Majoris, as measured and reduced by the writer.

Potsdam velocities. A like solution was made using the normal velocities derived from the Ludendorff measures. The following results were obtained which are in close agreement with Ludendorff's determination of these same quantities as previously given in this paper:

$$\gamma = -12.55 \pm 0.58 \text{ km.,}$$

$$\frac{m_1}{m_2} = 1.001 \pm 0.021.$$

* *Publications of the Allegheny Observatory*, Vol. I, No. 22, page 175.

This value of $\frac{m_1}{m_2}$ and its probable error throw

some light on the fact that Ludendorff found from a part of his observations that the relative order of magnitude of this pair seemed to be reversed.

At this point it should be noted that there is less than 0.1 km. difference between this value of γ and the mean of the velocities obtained by Ludendorff for ζ_2 Ursæ Majoris.

Measures of plates showing single lines. These plates were measured and reduced with the same care as the plates showing the lines measurably separated. There were thirty-eight such spectrograms. In cases of this kind the measurement is that of the blends of the two sets of spectral lines. In case one spectrum had been decidedly stronger than the other, the fact should have revealed itself by an inclination of the line connecting the normal places formed from these velocities. No such inclination is evident, indicating that the spectra are of practically the same intensity. The apparent difference in the relative intensity of the spectral lines of the two components as noted by Miss Maury in the *Astrophysical Journal*, Vol. 8, page 174, was not observed in the measurement of the spectrograms made at this Observatory. The plates secured for the cross point at the steepest part of the curves were too few to give a good determination. However, the weights assigned to each of the other four normal places show the determinations to be of considerable importance in that they should give, theoretically, the center of mass velocity. The weighted mean of the last four normal places is -7.26 km.

The measures from the single line plates as combined into normal places are given below. The table shows the phase, limits of phase, weight, velocity and residual. These normal velocities are shown by the small dotted circles in Plate VII.

SUMMARY OF RESULTS

That a comparison of the elements may be made and the differences which occur may be clearly evident, the data concerning each component as found from both sets of measures are brought together. Rather unexpected differences in the values of some of the elements are found. These apparent discrepancies will be discussed later.

TABLE VII. VELOCITIES.
(FROM MEASURES OF SINGLE LINES.)

ANN ARBOR OBSERVATIONS

NO.	PHASE.	LIMITS OF PHASE.	WT.	SINGLE LINES.	
				VELOCITY.	RESIDUAL.
(1)	(2)	(3)	(4)	(5)	(6)
1	5.523	4.90 to 5.36	10	-4.92	+2.34
2	12.564	12.05 to 13.00	82	-6.99	+0.27
3	14.388	13.82 to 15.14	52	-8.56	-1.30
4	16.458	15.97 to 17.05	58	-8.56	-1.30
5	18.458	17.60 to 18.95	33	-5.48	+1.78

CORRECTED ELEMENTS OF COMPONENT 1.

LUDENDORFF.	HADLEY.
$P = 20.53644$ days,	20.53644 days,
$\mu = 17^{\circ}.5298$,	$17^{\circ}.5298$,
$T = 5.6058 \pm 0.0700$ days,*	5.8426 ± 0.0772 days,*
$e = 0.5315 \pm 0.0102$,	0.5178 ± 0.0081 ,
$\omega = 103^{\circ} 52'.2 \pm 2^{\circ} 47'.9$,	$101^{\circ} 43'.6 \pm 2^{\circ} 24'.9$,
$K = 66.05 \pm 0.77$ km.,	71.65 ± 0.69 km.,
$\gamma = -11.67$ km.,	-6.22 km.,
$a \sin i = 15,800,000$ km.,	$17,310,000$ km.,
$m_1 = \frac{1.57 \odot}{\sin^3 i}$.	$\frac{1.83 \odot}{\sin^3 i}$.

* From 1912, March 10.000.

CORRECTED ELEMENTS OF COMPONENT 2.

$P = 20.53644$ days,	20.53644 days,
$\mu = 17^{\circ}.5298$,	$17^{\circ}.5298$,
$T = 5.4515 \pm 0.0835$ days,*	5.9461 ± 0.0582 days,*
$e = 0.5533 \pm 0.0124$,	0.5447 ± 0.0068 days,
$\omega = 276^{\circ} 55'.8 \pm 2^{\circ} 17'.5$,	$288^{\circ} 42'.5 \pm 1^{\circ} 58'.6$,
$K = 67.76 \pm 0.98$ km.,	69.21 ± 0.56 km.,
$\gamma = -11.96$ km.,	-9.35 km.,
$a \sin i = 15,940,000$ km.,	$16,390,000$ km.,
$m_2 = \frac{1.46 \odot}{\sin^3 i}$,	$\frac{1.79 \odot}{\sin^3 i}$,
$\frac{m_1}{m_2} = 1.079$.	1.025 .

* From 1912, March 10.000.

A comparison of the four sets of elements is both interesting and instructive. The differences between the values of certain of the quantities is greater than the probable errors would lead one

to expect. It seemed advisable to derive the elements in this way so that such differences as do occur may be clearly evident. These differences in no way discount the accuracy of the elements of such an orbit in so far as they are to represent the observations, but they do raise the question as to whether the observations give the exact conditions of motion as they exist in the binary system. However, as has been pointed out, the measures in this case were difficult.

The discrepancies between the four sets of elements, which have been tabulated, may be divided into two classes: those between the elements for the two components at the same epoch and those between the elements of the same component at different epochs.

Relatively large differences of both kinds were to be expected in the case of this star on account of the limitations imposed by the character of the spectra and by the existence of two components for each line when relative motion was sufficiently great to produce separation. Vogel has called attention to the poor quality of the lines from the standpoint of measurability. However, his conclusions are not surprising. In general, when resolved, each line of the star's spectrum falls upon the continuous spectrum of the other star and thus cannot present a clearly defined interruption. In other cases, a line of one spectrum might also coincide with a very faint line of the other spectrum and thus render the measure defective. Further, during about half of each orbital cycle the relative motion of the two components was too small to cause complete separa-

tion of the lines, hence the measures under those conditions were of uncertain value. From a consideration of these facts it becomes very evident that the available data for the determination of elements is unavoidably limited in this case.

In a system of this character, with high orbital eccentricity and relatively close approach of the two bodies at periastron, there would be strong probability that librations and tidal disturbances would arise to such an extent that they would produce irregularities in the velocity curves. These irregularities in the velocity curves would in turn give rise to spurious differences in the elements of the two orbits. Irregularities, which are probably of this nature, have been observed in highly eccentric orbits with relatively short periods, such as θ Aquilæ and ψ Orionis. The great variation in the rate of anomalistic motion as the bodies pass from apastron to periastron in very eccentric orbits suggests at once a cause for at least a part of the apparent lack of consistency in the results.

An examination of the original papers discussing binaries with highly eccentric orbits shows clearly that the difficulties which have been encountered in the study of this star are not peculiar to it alone. The writer has examined with some care the papers dealing with thirty-three orbits whose eccentricities exceed 0.3, the average value being 0.498. While it is difficult, if not impossible, to determine what precision may be expected in such work, it is the opinion of the writer that not less than two-thirds of the cases examined show some peculiarity. In six cases secondary oscillations had been assumed in order to reduce the differences between observed and computed velocities. In many other cases a like treatment would undoubtedly have reduced the residuals. At least half of the cases examined show more or less marked indications of a secondary oscillation. Other cases showed unexpectedly large residuals. It seems more reasonable to attribute these irregularities in the observed velocities either directly or indirectly to the high eccentricity than it does to assume the presence of a small third body. There is no reason to believe that the presence of a small third body tends to produce a highly eccentric orbit. Visual observations have rarely revealed three-

body systems, indicating that they are few in number. It seems to be a reasonable supposition that the same is true of systems less widely separated which can be revealed only by spectroscopic methods of observing. Hence, it seems more to the point to attribute slight departures from elliptic motion to tidal disturbances present in highly eccentric orbits.

In this connection, attention should be called to two sets of elements as derived from θ Aquilæ by Baker and Harper. A discussion of the elements of the orbit of this binary is to be found in the *Publications of the Royal Astronomical Society of Canada*, Vol. III, page 95. The difference in time for the two sets of observations is too small for the elements to have suffered appreciable change. The lack of agreement between the values as determined for certain elements is of the same order as has been found in the study of ζ_1 Ursæ Majoris. In both cases the probable errors indicate that the elements as derived satisfy the observations well, but the differences between the two sets of elements for the same star seem to show that the actual conditions of motion in the orbit are not obtained with as great precision as the probable errors of a single set of elements indicate.

A careful examination of the two plates showing the velocity curves for ζ_1 Ursæ Majoris reveals irregularities which can hardly be attributed to accidental errors. The many cases in which the observed velocities for the two sets of measures at approximately the same phase are either both larger or both smaller than the corresponding computed velocities is striking. The frequency with which this occurs is convincing evidence of its reality even though the real cause can not be definitely stated.

Apparently real irregularities existing in the velocity curves of stars of certain classes may place a limit on the accuracy of our determination of their orbital elements. The discrepancies observed in this paper furnish some evidence as to the uncertainty which may come from such irregularities and in that connection are highly instructive.

An effort was made to combine the four sets of elements, first, using the elements from the

two components at the same epoch, and then using these elements in securing the final values.

Elements from the Potsdam Observations. In bringing together the values of the elements as derived from the two components and also from the two sets of observations, weights will be assigned which are inversely proportional to the squares of the probable errors of the different determinations.

Nothing further need be said concerning the values of P and μ .

Evidently, the two values of T cannot be different since the two stars of the system must both have the same time for periastron passage; hence, the weighted mean will be taken as the best value for this quantity.

In connection with the value of e , it is of interest to note that the eccentricity of the orbit of Component 2 is the larger for both sets of measures. The weighted mean will be accepted as the best determination of this quantity since the two orbits cannot differ in this respect.

The two values of ω should differ by 180° . The best value of ω for Component 1 is probably given by taking the weighted mean of the two after 180° has been subtracted from the value of ω for Component 2.

The values of K for the two components are not very different. Only a small difference would be expected for a system in which $\frac{m_1}{m_2}$ is nearly equal to unity, as it is in this case.

Values for the ratio of the masses and for the center of mass velocity were obtained from the least square solution along with the other elements. The values of these two quantities alone were determined from other considerations by a least square solution. The means of these values will be taken as the most probable values of γ and $\frac{m_1}{m_2}$.

The two values for $a \sin i$ should not be the same unless the ratio of the masses be unity, which it is not. However, the difference should be small in the case of this star.

Elements from Ann Arbor Observations. The treatment will be like that of the Potsdam observations, the one exception being in connection with the center of mass velocity. A determination of the velocity of the center of mass was obtained from the plates showing single lines as has been mentioned already. The precision of this method is less than that of the other methods, hence, the weight assigned to the result will be only half as great as that of the other determinations.

It should be noted at this point that the value of $a \sin i$ is larger for Component 1 than for Component 2, whereas m_1 is greater than m_2 . The explanation is to be found in the fact that the two velocity curves as plotted from the normal velocities gave different values for the eccentricity. In the case of the Potsdam elements, the value of $a \sin i$ for Component 2 is the larger; however, it is only slightly larger, the difference being less than the value of $\frac{m_1}{m_2}$ would lead one to expect.

The elements of the two components for each set of observations having been combined in the manner set forth, the following sets of elements are given as the result of that process:

POTSDAM OBSERVATIONS.

$$P = 20.53644 \text{ days,}$$

$$\mu = 17^\circ.5298,$$

$$T = 5.5356 \pm 0.0546 \text{ days,*}$$

$$e = 0.54058 \pm 0.00805,$$

$$\omega = 99^\circ 39'.3 \pm 1^\circ 50'.9,$$

$$K_1 = 66.05 \pm 0.77 \text{ km.,}$$

$$K_2 = 67.76 \pm 0.98 \text{ km.,}$$

$$\gamma = -12.06 \text{ km.,}$$

$$m_1 = \frac{1.57 \odot}{\sin^3 i},$$

$$m_2 = \frac{1.16 \odot}{\sin^3 i},$$

$$\frac{m_1}{m_2} = 1.038,$$

$$a_1 \sin i = 15,800,000 \text{ km.}$$

$$a_2 \sin i = 15,940,000 \text{ km.}$$

* From 1912, March 10.000 G. M. T.

ANN ARBOR OBSERVATIONS.

$$\begin{aligned}
 P &= 20.53644 \text{ days,} \\
 \mu &= 17^\circ.5298, \\
 T &= 5.9088 \pm 0.0483 \text{ days,*} \\
 e &= 0.53248 \pm 0.00530, \\
 \omega &= 105^\circ 35'.9 \pm 1^\circ 33'.8, \\
 K_1 &= 71.65 \pm 0.69 \text{ km.,} \\
 K_2 &= 69.21 \pm 0.56 \text{ km.,} \\
 \gamma &= -7.22 \text{ km.,}
 \end{aligned}$$

$$m_1 = \frac{1.83 \odot}{\sin^3 i},$$

$$m_2 = \frac{1.79 \odot}{\sin^3 i},$$

$$\frac{m_1}{m_2} = 1.027,$$

$$a_1 \sin i = 17,310,000 \text{ km.,}$$

$$a_2 \sin i = 16,390,000 \text{ km.}$$

* From 1912, March 10.000, G. M. T.

CONCLUSIONS.

By combining the two sets of elements in the same manner as were the elements of the two components for one set of observations, we obtain the following as the most probable values and announce them as the definitive elements of ζ_1 Ursæ Majoris:

DEFINITIVE ELEMENTS.

$$\begin{aligned}
 P &= 20.53644 \text{ days,} \\
 \mu &= 17^\circ.5298, \\
 T &= 1912, \text{ March. } 15.7440 \pm 0.0364 \text{ days,} \\
 &\quad \text{G. M. T.,} \\
 e &= 0.53476 \pm 0.00482, \\
 \omega &= 103^\circ 57'.5 \pm 1^\circ 12'.4, \\
 K_1 &= 69.22 \pm 0.52 \text{ km.,} \\
 K_2 &= 68.83 \pm 0.56 \text{ km.,} \\
 \gamma &= -9.64 \text{ km.,}
 \end{aligned}$$

$$m_1 = \frac{1.70 \odot}{\sin^3 i},$$

$$m_2 = \frac{1.62 \odot}{\sin^3 i},$$

$$\frac{m_1}{m_2} = 1.032,$$

$$a_1 \sin i = 16.4 \text{ million km.,}$$

$$a_2 \sin i = 16.4 \text{ million km.}$$

There has been no marked change in the spectrum of the star so far as the writer was able to judge.

The small difference in the position of the line of nodes is not thought to be real. It is probably to be attributed to the peculiarities of the velocity curve, the positions of the normal places, and to personal equation.

An appreciable change in the eccentricity is improbable and the difference for the two components cannot be real so far as the actual orbits are concerned.

The values of the K 's for the two sets of observations are in fair agreement. No real change in these quantities is probable in so brief a period of years. Personal equation and choice of spectral lines would account for a part, if not all, of the difference. The half-amplitude of the velocity curve for Component 1 is slightly larger than it is for Component 2, whereas m_1 is greater than m_2 . This is due to peculiarities of the curves and the close approach of the ratio of the masses to unity.

The velocity of the center of mass of ζ_1 Ursæ Majoris and the mean velocities of ζ_2 Ursæ Majoris as determined from the Potsdam measures are in close agreement. The same is true of the Ann Arbor observations. It follows that the difference between the Potsdam and Ann Arbor values of γ for ζ_1 Ursæ Majoris is approximately equal to the difference in the mean velocity of ζ_2 Ursæ Majoris as determined at these two observatories. That each of the two stars should have a variable center of mass velocity and that the changes should be almost the same in magnitude and of the same sign seems to be an unreasonable assumption. Hence the writer has been led to the conclusion that the apparent change in γ for ζ_1 Ursæ Majoris is not real, but that it is due to systematic differences attributable to personal equation and other causes. The difference in this quantity noticed by Ludendorff as mentioned early in this paper may be explained by the fact that the parts of the spectrum used by Ludendorff in the two groups of plates were not identical.

The masses of the two components are nearly equal, but a summary of the data bearing on this

point indicates that one component is approximately three per cent. larger than the other.

The determinations of $a \sin i$ for the two components are somewhat uncertain, but are of nearly equal value. Probably 16.4 million km. for each component is as definite as the data warrant.

The positions of the normal places as shown in connection with the velocity curves plotted from the final elements indicate the presence of a disturbing factor. The reality of these irregularities is shown by the fact that in many cases the residuals for the normal places of practically the same phase for the two components are of the same sign. A secondary oscillation suggests itself. However, the unavoidable lack of data in this case covering nearly half of the length of the curves makes the study of such an oscillation very difficult. The examination of many orbits of high eccentricity has led the writer to the conclusion that irregularities are the rule in such cases rather than the exception, and that they are attributable directly or indirectly to the high eccentricity.

While the probable errors seem to show the precision with which the derived elements represent the observed velocities, they do not show the precision with which the elements represent

the actual conditions of motion of the star in its orbit.

The generally accepted theory as to the mutual increase of eccentricity and period seems to be sustained by the values of these two quantities as found in connection with this spectroscopic pair.

If the views as expressed by Wicksell in his paper entitled "Contributions to the Statistics of Spectroscopic Binary Stars" be correct, then this star would seem to belong to the class consisting of those binaries which have originated through fission rather than through the "capture process". The length of the period and the high eccentricity indicate that the existence of this star as a binary has been of long duration even though the spectral types of the two components show that they have not progressed very far in their evolutionary course.

The writer wishes to express his thanks to Professor Ralph H. Curtiss whose counsel, direction, and assistance have made this investigation possible. A word of appreciation is due Mr. Lewis Mellor of this Observatory for assistance in making plates.

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